Hybrid RRT: Motion Planning for Hybrid Dynamical Systems

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1. Motivation: robotics motion planning
1. Motivation: robots with hybrid dynamics

The motion planning problem for hybrid systems is to find a trajectory of states and inputs that starts from the initial state (set), ends within the final state (set), and satisfies both continuous and discrete dynamics and safety criterion.
Outline

1. Motivation

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2. Preliminaries on Hybrid System Modeling

A hybrid system $\mathcal{H}$ with inputs is modeled by a system of differential and difference equations as

$$\mathcal{H} : \begin{cases} \dot{x} = f(x, u) & (x, u) \in C \\
x^+ = g(x, u) & (x, u) \in D \end{cases}$$ (1)

where $x \in \mathbb{R}^n$ is state, $u \in \mathbb{R}^m$ is input,

- $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the flow map;
- $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the jump map;
- $C \subset \mathbb{R}^n \times \mathbb{R}^m$ is the flow set;
- $D \subset \mathbb{R}^n \times \mathbb{R}^m$ is the jump set.
A motivating example: A classic example of hybrid system is bouncing ball system. Consider a ball bouncing on a fixed horizontal surface. The surface is capable of affecting the velocity of the ball after the impact through control actions.
2. Preliminaries on Hybrid System Modeling

A motivating example: A classic example of hybrid system is bouncing ball system. Consider a ball bouncing on a fixed horizontal surface. The surface is capable of affecting the velocity of the ball after the impact through control actions.

- $x_1 =$ height, $x_2 =$ velocity
- $\gamma =$ the gravity constant, $\lambda =$ coefficient of restitution

$$
\dot{x} = \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} =: f(x, u) \quad \forall (x, u) \in C
$$

$$
x^+ = \begin{bmatrix} x_1 \\ -\lambda x_2 + u \end{bmatrix} =: g(x, u) \quad \forall (x, u) \in D
$$

where

$$
C := \{(x, u) \in \mathbb{R}^2 \times \mathbb{R} : x_1 \geq 0\}
$$

$$
D := \{(x, u) \in \mathbb{R}^2 \times \mathbb{R} : x_1 = 0, x_2 \leq 0, u \geq 0\}.
$$
2. Preliminaries on Hybrid System Modeling

**Hybrid time domain:** The solutions and inputs to a hybrid system $\mathcal{H}$ are parameterized by $(t, j)$ where
- $t \in \mathbb{R}_{\geq 0}$ denotes the normal time variable
- $j \in \mathbb{N}$ denotes the number of jumps.

**Hybrid time domains:**

$$\text{dom } \psi := ([0, t_1]) \times \{0\} \cup ([t_1, t_2]) \times \{1\} \cup \ldots \cup ([t_j, t_{j+1}]) \times \{j\} \cup \ldots$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_{J+1}$.
Definition 2.1 (Solution pair to a hybrid system (\texttt{altin2019hybrid}))

Given a pair of functions $\phi: \text{dom} \phi \to \mathbb{R}^n$ and $u: \text{dom} u \to \mathbb{R}^m$ defined on hybrid time domains, $(\phi, u)$ is a solution pair to hybrid system $\mathcal{H} = (C, f, D, g)$ if:

1. During flows,

\[ (\phi(t, j), u(t, j)) \in C \quad \dot{\phi}(t, j) = f(\phi(t, j), u(t, j)) \]
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   $$(\phi(t, j), u(t, j)) \in C \quad \dot{\phi}(t, j) = f(\phi(t, j), u(t, j))$$

2. At jumps,

   $$(\phi(t, j), u(t, j)) \in D \quad \phi(t, j + 1) = g(\phi(t, j), u(t, j)).$$
3. Problem Formulation

Problem 1 (Motion planning problem for hybrid systems)

Given

1. a hybrid system $\mathcal{H} = (C, f, D, g)$ with input $u \in \mathbb{R}^m$, state $x \in \mathbb{R}^n$;

find a pair $(\phi, u) : \text{dom}(\phi, u) \rightarrow \mathbb{R}^n \times \mathbb{R}^m$, namely, a motion plan, such that:

1. $(\phi, u)$ is a solution pair to $\mathcal{H}$;
3. Problem Formulation

Problem 1 (Motion planning problem for hybrid systems)

Given

1. a hybrid system $\mathcal{H} = (C, f, D, g)$ with input $u \in \mathbb{R}^m$, state $x \in \mathbb{R}^n$;
2. initial state set $X_0 \subset \mathbb{R}^n$;

find a pair $(\phi, u) : \text{dom}(\phi, u) \rightarrow \mathbb{R}^n \times \mathbb{R}^m$, namely, a motion plan, such that:

1. $(\phi, u)$ is a solution pair to $\mathcal{H}$;
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1. a hybrid system $\mathcal{H} = (C, f, D, g)$ with input $u \in \mathbb{R}^m$, state $x \in \mathbb{R}^n$;
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3. final state set $X_f \subset \mathbb{R}^n$;

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1. $(\phi, u)$ is a solution pair to $\mathcal{H}$;
2. $\phi(0, 0) \in X_0$;
3. there exists $(T, J) \in \text{dom}(\phi, u)$ such that $\phi(T, J) \in X_f$;
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Problem 1 (Motion planning problem for hybrid systems)

Given

1. a hybrid system $\mathcal{H} = (C, f, D, g)$ with input $u \in \mathbb{R}^m$, state $x \in \mathbb{R}^n$;
2. initial state set $X_0 \subset \mathbb{R}^n$;
3. final state set $X_f \subset \mathbb{R}^n$;
4. unsafe set $X_u \subset \mathbb{R}^n \times \mathbb{R}^m$;

find a pair $(\phi, u) : \text{dom}(\phi, u) \rightarrow \mathbb{R}^n \times \mathbb{R}^m$, namely, a motion plan, such that:

1. $(\phi, u)$ is a solution pair to $\mathcal{H}$;
2. $\phi(0, 0) \in X_0$;
3. there exists $(T, J) \in \text{dom}(\phi, u)$ such that $\phi(T, J) \in X_f$;
4. $(\phi(t, j), u(t, j)) \notin X_u$ for each $(t, j) \in \text{dom}(\phi, u), t + j \leq T + J$. 
4. HyRRT Algorithm

4.1 Algorithm description

**Algorithm 1 HyRRT algorithm**

Input: $X_0, X_f, X_u, \mathcal{H} = (C, f, D, g), \mathcal{U} = (U_C, U_D), p_n \in (0, 1)$

1: $\mathcal{T}.\text{init}(X_0)$
2: for $i = 1$ to $k$ do
3: randomly select a real number $n$ from $[0, 1]$
4: if $n \leq p_n$ then
5: $x_{\text{rand}} \leftarrow \text{random}_\text{state}(C')$
6: extended $\leftarrow \text{extend}(\mathcal{T}, x_{\text{rand}}, \mathcal{U}, \mathcal{H}, X_u, \text{flow})$
7: else
8: $x_{\text{rand}} \leftarrow \text{random}_\text{state}(D')$
9: extended $\leftarrow \text{extend}(\mathcal{T}, x_{\text{rand}}, \mathcal{U}, \mathcal{H}, X_u, \text{jump})$
10: end if
11: if extended $== 1$ \\
12: & check\_solution(\mathcal{T}, X_0, X_f, C, \psi_{\text{sol}}) == 1$ then
13: return $\psi_{\text{sol}}$
14: end if
15: end for
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The propagation results:
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4.2 Probabilistic completeness

Definition 4.1

*(Probabilistic completeness (kleinbort2018probabilistic)) A sampling-based algorithm is said to be probabilistically complete if the probability of failing to find a solution is converging to $0$, as the number of samples approaches to infinity.*
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(Probabilistic completeness (kleinbort2018probabilistic)) A sampling-based algorithm is said to be probabilistically complete if the probability of failing to find a solution is converging to 0, as the number of samples approaches to infinity.

Theorem 1

The proposed HyRRT is probabilistically complete for any given motion planning problem for hybrid systems formulated as in Problem 1.
4. HyRRT Algorithm

4.3 Simulation results: Algorithm 1 leads to a HyRRT software tool\(^1\) to solve the motion planning problems for hybrid systems. The simulation is implemented in MATLAB software and processed by a 2.2 GHz Intel Core i7 processor. The simulation takes 0.34 seconds to finish.

\(^1\)Code at https://github.com/HybridSystemsLab/hybridRRT.
4. HyRRT Algorithm

4.3 Simulation results: The simulation is implemented in MATLAB software and processed by a 3.5 GHz Intel Core i5 processor. The simulation takes 57.6 seconds to finish.

![Graphs showing state transitions over time]
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Table: Computation Costs of HyRRT and FBP in the Biped Example.

<table>
<thead>
<tr>
<th></th>
<th>Time Consumption (seconds)</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>HyRRT</td>
<td>57.6</td>
<td>2357</td>
</tr>
<tr>
<td>FBP</td>
<td>1608.2</td>
<td>3796</td>
</tr>
</tbody>
</table>
Acknowledgement

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Thank you for your attention. Any questions?